第九次作业

Zstar

10.

(1).

$$f(\theta,\varphi)=\sin^2\theta\cos^2\varphi=\frac{1-\cos 2\theta}{2}\frac{1+\cos 2\varphi}{2}=\frac{1}{3}-\frac{1}{3}P_2^0(\cos\theta)+\frac{1}{6}P_2^2(\cos\theta)\cos 2\varphi$$

(2).

$$f(\theta,\varphi)=(1+3\cos\theta)\sin\theta\cos\varphi=P_1^1(\cos\theta)\cos\varphi+P_2^1(\cos\theta)\cos\varphi$$

11.

(1).参考书上式**(3.3.25)**

$$u(r,\theta,\varphi)=\sum_{n=0}^{+\infty}\sum_{m=0}^{n}(A_nr^n+B_nr^{-(n+1)})P_n^m(\cos\theta)(C_{nm}\cos m\varphi+D_{nm}\sin m\varphi)$$

球内$B_n=0$

$$u|_{r=a}=\sin 2\theta\cos\varphi=\frac{2}{3}P_2^1(\cos\theta)\cos\varphi$$

比较系数可得

$$u=\frac{2}{3}(\frac{r}{a})^2P_2^1(\cos\theta)\cos\varphi$$

(2). $A_n=0$, 直接按课本方法做即可

$$u=\sum_{n=0}^{+\infty}\sum_{m=0}^{n}\frac{-a^{n+2}}{(n+1)r^{n+1}}(A_{nm}\cos m\varphi+B_{nm}\sin m\varphi)\,P_n^m(\cos\theta)$$

$$A_{nm}=\frac{(n-m)!}{(n+m)!}\frac{(2n+1)}{2\delta_m\pi}\int_0^{2\pi}\cos m\varphi\mathrm{d}\varphi\int_0^{\pi}f(\theta,\varphi)P_n^m(\cos\theta)\sin\theta\mathrm{d}\theta$$

$$B_{nm}=\frac{(n-m)!}{(n+m)!}\frac{(2n+1)}{2\pi}\int_0^{2\pi}\sin m\varphi\mathrm{d}\varphi\int_0^{\pi}f(\theta,\varphi)P_n^m(\cos\theta)\sin\theta\mathrm{d}\theta$$

$$\delta_m=\begin{cases}1, & m\neq 0\\ 2, & m=0\end{cases}$$

12.

(1).

$$\int x^2J_0(x)dx=\int xd(xJ_1(x))=x^2J_1(x)-\int xJ_1(x)dx=x^2J_1(x)+xJ_0(x)-\int J_0(x)dx$$

(2).

$$\int x^4 J_1(x)dx = \int x^2 \cdot x^2 J_1(x)dx = x^4 J_2 - 2\int x^3 J_2 dx = x^4 J_2 - 2x^3 J_3 + C$$
$$= x^4 J_2 - 2x^3[4x^{-1}J_2 - J_1] + C$$

再代入：$J_2 = 2x^{-1}J_1 - J_0$ 化简为：

$$x^2(8-x^2)J_0 + 4x(x^2-4)J_1 + C$$

(3).

$$\int J_3 dx = \int (J_1 - 2J_2')dx = -J_0 - 2J_2 + C = J_0 - 4x^{-1}J_1 + C$$

(4).

$$\int xJ_1 dx = -xJ_0 + \int J_0 dx$$

13.

$$f(x) = \sum_{n=1}^{+\infty} c_n J_1(\omega_n x), c_n = \frac{1}{||J_1(\omega_n x)||^2}\int_0^1 x^2 J_1(\omega_n x)dx$$

注意固有函数系$J_\nu(w_{1n}r), J_\nu(w_{2n}r), J_\nu(w_{3n}r)$分别是满足相应边界条件的函数空间里的**权为r的完备正交函数系**, 同时也要知道对应于三类边界条件固有函数系的模（p118）

$$c_n = \frac{2}{J_2^2(\omega_n)}\int_0^1 x^2 J_1(\omega_n x)dx = \frac{2}{J_2^2(\omega_n)} \cdot \frac{J_2(\omega_n)}{\omega_n} = \frac{2}{\omega_n J_2(\omega_n)}$$

即得：

$$f(x) = \sum_{n=1}^{+\infty} \frac{2}{\omega_n J_2(\omega_n)} J_1(\omega_n x),$$

14.

$$f(x) = \sum_{n=1}^{+\infty} c_n J_0(\omega_n x), c_n = \frac{1}{||J_0(\omega_n x)||^2}\int_0^1 xJ_0(\omega_n x)dx = \frac{1}{2J_1^2(2\omega_n)} \cdot \frac{J_1(\omega_n)}{\omega_n}$$
$$\Longrightarrow f(x) = \sum_{n=1}^{+\infty} \frac{J_1(\omega_n)}{2J_1^2(2\omega_n)\omega_n} J_0(\omega_n x)$$

15.

(1). 先分离变量得：

$$\begin{cases} (rR')' + \lambda rR = 0, 0 < r < l \\ |R(0)| < +\infty, R(l) = 0 \end{cases} \text{和} T'' + 2hT' + \lambda a^2 T = 0$$

其中关于R的方程为第I类边界条件0阶Bessel方程，得：

$$\text{固有值：} \lambda_n = \omega_{1n}^2, \omega_{1n}\text{是}J_0(\omega l) = 0\text{的第}n\text{个正根，固有函数}R_n(r) = J_0(\omega_{1n}r)$$

再代入T的方程:

$$T''+2hT'+\omega_{1n}^2a^2T=0\Longrightarrow T_n(t)=e^{-ht}(A_n\cos q_nt+B_n\sin q_nt),\quad q_n=\sqrt{\omega_{1n}^2a^2-h^2}$$

$$u(t,r)=\sum_{n=1}^{+\infty}T_n(t)R_n(r)=\sum_{n=1}^{+\infty}e^{-ht}(A_n\cos q_nt+B_n\sin q_nt)J_0(w_{1n}r)$$

最后利用边界条件:

$$u(0,r)=\sum_{n=1}^{+\infty}A_nJ_0(w_{1n}r)=\varphi(r)\Longrightarrow A_n=\frac{2}{l^2J_1^2(w_{1n}l)}\int_0^l r\varphi(r)J_0(\omega_{1n}r)dr$$

$$u_t(0,r)=-hu(0,r)+\sum_{n=1}^{+\infty}q_nB_nJ_0(\omega_{1n}r)=0\Longrightarrow B_n=\frac{h}{q_n}A_n$$

综合起来即得答案

(3).先分离变量:

$$\begin{cases}(xX')'+\lambda xX=0,0<x<1\\|X(0)|<+\infty,X(1)=0\end{cases}\text{和}T''+(\lambda+b^2)T=0$$

关于X的方程仍是第I类边界条件0阶Bessel方程，得:

$$\text{固有值：}\lambda_n=\omega_{1n}^2,\omega_{1n}\text{是}J_0(\omega)=0\text{的第}n\text{个正根，固有函数}R_n(r)=J_0(\omega_{1n}x)$$

再代入T的方程:

$$T''+(\omega_{1n}^2+b^2)T=0\Longrightarrow T_n(t)=A_n\cos q_nt+B_n\sin q_nt,\ q_n=\sqrt{w_{1n}^2+b^2}$$

$$u(t,r)=\sum_{n=1}^{+\infty}T_n(t)X_n(x)=\sum_{n=1}^{+\infty}(A_n\cos q_nt+B_n\sin q_nt)J_0(w_{1n}x)$$

最后利用边界条件:

$$u(0,x)=0\Longrightarrow A_n=0$$

$$u_t(0,x)=\sum_{n=1}^{+\infty}q_nB_nJ_0(\omega_{1n}x)=\psi(x)\Longrightarrow B_n=\frac{2}{J_1^2(w_{1n})q_n}\int_0^1 x\psi(x)J_0(\omega_{1n}x)dx$$

综合即得答案

16.

先写出定解问题:

$$\begin{cases}\frac{\partial u}{\partial t}=a^2\Delta_3u,0\le r<R,-\infty<z<+\infty\\u|_{r=R}=u_0,u|_{t=0}=0\end{cases}$$

令$u=v+u_0$, 求解关于v的方程，v具有旋转对称性且与z无关，令$v=X(r)T(t)$

分离变量得:

$$T'+\lambda a^2T=0\text{和}\begin{cases}(rX')'+\lambda rX=0,0<r<R\\X(0)<+\infty,X(R)=0\end{cases}$$

这利用X主要是为了与边界R区分,由此得:

固有值：$\lambda_n=\omega_{1n}^2,\ \omega_{1n}$是$J_0(\omega_nR)=0$的第$n$个正根，固有函数：$X_n(r)=J_0(\omega_{1n}r),n=1,2,\ldots$

代入T的方程:

$$T_n(t)=C_ne^{-\omega_{1n}^2a^2t}$$

$$v(t,r)=\sum_{n=1}^{+\infty}T_n(t)X_n(r)=\sum_{n=1}^{+\infty}C_ne^{-\omega_{1n}^2a^2t}J_0(\omega_{1n}r)$$

利用边界条件:

$$v(0,r)=\sum_{n=1}^{+\infty}C_nJ_0(\omega_{1n}r)=-u_0\Longrightarrow C_n=-\frac{2u_0}{RJ_1(\omega_{1n}R)\omega_{1n}}$$

$$u=v+u_0=u_0-\frac{2u_0}{R}\sum_{n=1}^{+\infty}\frac{1}{J_1(\omega_{1n}R)\omega_{1n}}e^{-\omega_{1n}^2a^2t}J_0(\omega_{1n}r)$$

关于答案给出的形式，可以这样得到:

$$\begin{cases}R^2(rX')'+\lambda R^2rX=0,0<r<R\\X(0)<+\infty,X(R)=0\end{cases}$$

此时的固有值变为$\lambda_n=\frac{\omega_n^2}{R^2}$,令$t=\omega\frac{r}{R},y(t)=X(\frac{tR}{\omega})$得：$t^2y''+ty'+t^2y=0$仍为Bessel方程，此时的解为:

$$\lambda_n=\frac{\omega_n^2}{R^2},w_n\text{为}J_0(\omega_n)=0\text{的第}n\text{个正根，固有函数为：}R_n(r)=J_0(t)=J_0(\frac{\omega_nr}{R})$$

17.

定解问题:

$$\begin{cases}\Delta_3u=0\\u_r|_{r=a}=0\\u|_{z=0}=f_1(r),u|_{z=h}=f_2(r)\end{cases}$$

令$u=R(r)Z(z)$分离变量:

$$\begin{cases}(rR')'+\lambda rR=0,0<r<a\\|R(0)|<+\infty,R'(a)=0\end{cases}\text{和}Z''-\lambda Z=0$$

有S-L定理知:

$$\text{固有值为：}\lambda=\omega_n^2,\omega_0=0,w_n\text{为}J_0'(w_na)=0\text{的第}n\text{个正根}(for\ n>1)$$
$$\text{固有函数：}R_0(r)=1,R_n(r)=J_0(\omega_nr),for\ n>1$$

代入Z的方程得:

$$Z_0(z) = C_0 + D_0 z,\ Z_n(z) = \begin{Bmatrix} ch\omega_n z \\ sh\omega_n z \end{Bmatrix}$$

$$u(r,z) = C_0 + D_0 z + \sum_{n=1}^{+\infty} (C_n ch\omega_n z + D_n sh\omega_n z) J_0(\omega_n r)$$

$$u|_{z=0} = C_0 + \sum_{n=1}^{+\infty} C_n J_0(\omega_n r) = f_1(r)$$

$$u|_{z=h} = C_0 + D_0 h + \sum_{n=1}^{+\infty} (C_n ch\omega_n h + D_n sh\omega_n h) J_0(\omega_n r) = f_2(r)$$

得系数：

$$C_0 = \frac{2}{a^2} \int_0^a f_1(r) r dr \triangleq f_{10}$$

$$C_n = \frac{2}{a^2 J_0^2(\omega_n a)} \int_0^a J_0(\omega_n r) r f_1(r) dr \triangleq f_{1n}$$

$$C_0 + D_0 h = \frac{2}{a^2} \int_0^a f_2(r) r dr \triangleq f_{20}$$

$$C_n ch\omega_n h + D_n sh\omega_n h = \frac{2}{a^2 J_0^2(\omega_n a)} \int_0^a J_0(\omega_n r) r f_2(r) dr \triangleq f_{2n}$$

$$D_0 = \frac{f_{20} - f_{10}}{h},\ D_n = \frac{f_{2n} - f_{1n} ch\omega_n h}{sh\omega_n h}$$