P_{132} . 2.

$$xy'' + (1-x)y' + \lambda y = 0$$

设 $y(x) = x^{\rho}\sum_{n=0}^{\infty} a_n x^n$， $y'' = \sum_{n=0}^{\infty} a_n (n+\rho)(n+\rho-1)x^{n+\rho-2}$ $y' = \sum_{n=0}^{\infty} a_n (n+\rho) x^{n+\rho-1}$.

$$xy'' = \sum_{n=0}^{\infty} a_n (n+\rho)(n+\rho-1)x^{n+\rho-1}, \quad (1-x)y' = \sum_{n=0}^{\infty} a_n(n+\rho)x^{n+\rho-1} - \sum_{n=0}^{\infty} a_n(n+\rho)x^{n+\rho}.$$

$$xy'' + (1-x)y' + \lambda y = \sum_{n=0}^{\infty} a_n[(n+\rho)(n+\rho-1) + (n+\rho)]x^{n+\rho-1} + \sum_{n=0}^{\infty} a_n[\lambda - (n+\rho)]x^{n+\rho}$$

$$= a_0\rho^2 x^{\rho-1} + \sum_{n=1}^{\infty} [a_n(n+\rho)^2 - a_{n-1}(n+\rho-1-\lambda)]x^{n+\rho-1} = 0.$$

可得， $a_0\rho^2 = 0$， $a_n = \dfrac{n+\rho-(\lambda+1)}{(n+\rho)^2} a_{n-1}$.

$$\Rightarrow \rho = 0, \quad a_n = \frac{n-(\lambda+1)}{n^2} a_{n-1} = \frac{(-1)(\lambda-n+1)}{n^2} a_{n-1}$$

即 $a_n = \dfrac{(-1)^n \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)}{(n!)^2} a_0$.

若取 $a_0 = 1$，则有

$$y(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-k+1)}{(k!)^2} x^k$$

当 $\lambda = n$，$(n = 0, 1, 2, \cdots)$ 时，

$$y(x) = L_n(x) = \sum_{k=1}^{n} (-1)^k C_n^k \frac{x^k}{k!} \text{ 为多项式.}$$

3.

$$P_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & n = 2k+1 \\ \dfrac{(-1)^k(2k)!}{2^{2k}(k!)^2} = \dfrac{(-1)^k(2k-1)!!}{(2k)!!}, & n = 2k \end{cases}$$ 其中 $k = 1, 2, 3, \cdots$.

$$P_n'(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k(2n-2k)!\,(n-2k)}{2^n k!(n-k)!(n-2k)!} x^{n-2k-1}$$

$$P_n'(0) = \begin{cases} 0, & n = 2k. \\ \dfrac{(-1)^k(2k+2)!}{2^{2k+1}k!(k+1)!} = \dfrac{(-1)^k(2k+1)!!}{(2k)!!}, & n = 2k+1 \end{cases}$$ 其中 $k = 0, 1, 2, \cdots$.