4.

> (1)
>
> $$\frac{\partial^2 u}{\partial x \partial y} + \frac{2}{y}\frac{\partial u}{\partial x} = 2x$$

设$f(x,y) = \frac{\partial u}{\partial x}$, 方程转化为

$$\begin{aligned}
\frac{\partial f}{\partial y} + \frac{2}{y}f = 2x \\
\frac{\partial}{\partial y}(fe^{2\ln y}) = 2e^{2\ln y} \\
\frac{\partial}{\partial y}(fy^2) = 2y^2 \\
f = \frac{2}{3}y + C_0(x)\frac{1}{y^2} \\
u = \frac{1}{3}x^2 y + C_1(x)\frac{1}{y^2} + C_2(y)
\end{aligned}$$

> (2)
>
> $$\frac{\partial u}{\partial y} + p(x,y)u = q(x,y)$$

按照讲义上解法即可

$$u(x,y) = e^{-\int pdy}[\int e^{\int pdy} qdy + C(x)]$$

12.

> (1)
>
> $$(y+z)u_x + (z+x)u_y + (x+y)u_z = 0$$

按讲义上解法即可.

$$u = \varphi\left(\frac{x-z}{y-z}, (x-y)^2(x+y+z)\right)$$

> (2)
>
> $$yu_x - xu_y = x^2 - y^2$$

特征方程

$$\frac{dx}{y} = -\frac{dy}{x}$$

得到首次积分

$$x^2 + y^2 = C$$

取变量代换$\xi = x^2 + y^2, \eta = xy$, 代回得到

$$\frac{\partial u}{\partial \eta} = 1 \Rightarrow u = \eta + f(\xi)$$

得到解

$$u = xy + f(x^2 + y^2)$$

13.

> (1)
>
> $$\begin{cases} \sqrt{x}u_x + \sqrt{y}u_y + \sqrt{z}u_z = 0 \quad x > 0, y > 0, z > 0 \\ u|_{x=1} = y - z \end{cases}$$

特征方程

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$

得到两个独立的首次积分

$$\sqrt{x} - \sqrt{y} = C \quad \sqrt{y} - \sqrt{z} = C$$

变量代换$\xi = \sqrt{x} - \sqrt{y}, \eta = \sqrt{x} - \sqrt{z}, \zeta = y$. 方程转化为

$$\frac{\partial}{\partial \zeta} u = 0$$

得到通解

$$u = f(\xi, \eta)$$

带入边界条件

$$f(1 - \sqrt{y}, 1 - \sqrt{z}) = y - z = (1 - \sqrt{y} - 1)^2 - (1 - \sqrt{z} - 1)^2$$

得到解为

$$\begin{aligned} u &= (\sqrt{x} - \sqrt{y} - 1)^2 - (\sqrt{x} - \sqrt{z} - 1)^2 \\ &= [2(\sqrt{x} - 1) - (\sqrt{y} + \sqrt{z})](\sqrt{z} - \sqrt{y}) \\ &= y - z - 2(\sqrt{x} - 1)(\sqrt{y} - \sqrt{z}) \end{aligned}$$

14.

> (2)
>
> $$\begin{gathered} u_{xx} + 2u_{xy} - 3u_{yy} = 0 \\ u(x, 0) = 3x^2, u_y(x, 0) = 0 \end{gathered}$$

方程改写为

$$(\partial_x + 3\partial_y)(\partial_x - \partial_y)u = 0$$

因此可以转写为

$$\begin{cases} (\partial_x + 3\partial_y)v = 0 \\ (\partial_x - \partial_y)u = v \end{cases}$$

两个方程的特征方程分别为

$$dx - \frac{1}{3}dy = 0 \quad dx + dy = 0$$

得到两个无关的首次积分作为新的自变量

$$\xi = x - \frac{1}{3}y \quad \eta = x + y$$

方程转化为

$$\frac{\partial}{\partial\xi\partial\eta}u=0$$

得到解

$$u=f(x-\frac{1}{3}y)+g(x+y)$$

带入边界条件

$$\begin{cases} f(x)+g(x)=3x^2 \\ -\frac{1}{3}f'(x)+g'(x)=0 \quad \Rightarrow f(x)-3g(x)=C \end{cases}$$

解得

$$\begin{cases} f(x)=\frac{9}{4}x^2+\frac{1}{4}C \\ g(x)=\frac{3}{4}x^2-\frac{1}{4}C \end{cases}$$

得到解为

$$u=\frac{9}{4}(x-\frac{1}{3}y)^2+\frac{3}{4}(x+y)^2$$

> (2)
>
> $$\begin{cases} u_{tt}=a^2u_{xx} \\ u|_{x-at=0}=\varphi(x) \\ u|_{x=0}=f(t) \end{cases}$$
>
> 若$x=0$上的条件改为$u_x|_{x=0}=f(t)$, 解有何变化?

通解

$$u=h(x-at)+g(x+at)$$

带入边界条件

$$\begin{cases} h(0)+g(x+at)=\varphi(x)\Rightarrow h(0)+g(2x)=\varphi(x) \quad x>0 \\ h(-at)+g(at)=f(t) \quad at>0 \end{cases}$$

得到

$$\begin{cases} g(x)=\varphi(\frac{1}{2}x)-h(0) \quad x>0 \\ h(x)=f(-\frac{x}{a})-g(-x)=f(-\frac{x}{a})-\varphi(-\frac{1}{2}x)+h(0) \quad x<0 \end{cases}$$

得到解

$$\begin{aligned} u&=h(x-at)+g(x+at) \quad x-at<0,x+at>0 \\ &=\varphi(\frac{x+at}{2})-\varphi(\frac{at-x}{2})+f(t-\frac{x}{a}) \end{aligned}$$

若改变条件, 则

$$h'(-at)+g'(at)=f(t)\Rightarrow -h(-at)+g(at)=a\int_0^t f(\xi)d\xi+h(0)+g(0)$$

得到

$$\begin{cases} g(x) = \varphi(\frac{1}{2}x) - h(0) & x > 0 \\ h(x) = \varphi(-\frac{1}{2}x) - a\int_0^{-\frac{x}{a}} f(\xi)d\xi - g(0) \end{cases}$$

由

$$h(0) = \varphi(0) - g(0) \Rightarrow h(0) + g(0) = \varphi(0)$$

得到解

$$u = \varphi(\frac{x+at}{2}) + \varphi(\frac{at-x}{2}) - \varphi(0) - a\int_0^{t-\frac{x}{a}} f(\xi)d\xi$$

15.

> 一端固定的半无界弦定解问题
>
> $$\begin{cases} u_{tt} = a^2 u_{xx} & t > 0, x > 0 \\ u(t,0) = 0, \\ u(0,x) = \sin x, u_t(0,x) = kx. \end{cases}$$
>
> 若改$u(0,x) = \cos x$, 则$u = ?$

解:

解要满足$u(t,0) = 0$, 需要进行奇延拓. 注意到边界条件本身就是奇函数, 直接带入达朗贝尔公式计算即可

$$\begin{aligned} u(t,x) &= \frac{1}{2}[\Phi(x-at) + \Phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\xi)d\xi \\ &= \frac{1}{2}[\sin(x-at) + \sin(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} k\xi d\xi \\ &= \sin x \cos at + kxt \quad x > 0 \end{aligned}$$

考虑边界条件$u(0,x) = \cos x$. 注意到我们进行延拓的目的是保证$u(t,0) = 0$, 而对于弦振动方程, 当初值条件都为奇函数时解对于x也是奇函数, 这自然满足边界条件的需求. 因此进行奇延拓

$$\Phi(x) = \begin{cases} \cos x & x > 0 \\ 0 & x = 0 \\ -\cos x & x < 0 \end{cases}$$

带入公式计算

$$u(t,x) = \frac{1}{2}[\Phi(x-at) + \Phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\xi)d\xi$$

我们只需要关注$x > 0$的部分即可

$$\begin{aligned} u(t,x) &= \frac{1}{2}[\Phi(x-at) + \Phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \Psi(\xi)d\xi \\ &= \begin{cases} \frac{1}{2}[\cos(x-at) + \cos(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} k\xi d\xi & x - at > 0 \\ \frac{1}{2a}\int_{x-at}^{x+at} k\xi d\xi & x - at = 0 \\ \frac{1}{2}[-\cos(x-at) + \cos(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} k\xi d\xi & x - at < 0 \end{cases} \\ &= \begin{cases} \cos(x)\cos(at) + kxt & x - at > 0 \\ kxt & x - at = 0 \\ -\sin(x)\sin(at) + kxt & x - at < 0 \end{cases} \end{aligned}$$