1. 证明下列等式

> (1)
>
> $$\delta(x)\cos x+\delta(x)\sin x=\delta(x)$$

$\forall\varphi(x)\in C(R)$

$$\int_{-\infty}^{\infty}dx\ [\delta(x)\cos x+\delta(x)\sin x]\varphi(x)=f(0)=\int_{-\infty}^{\infty}dx\ \delta(x)\varphi(x)$$

得证

> (2)
>
> $$x\delta'(x)=-\delta(x)$$

$$\begin{aligned}\int_{-\infty}^{\infty}dx\ x\delta'(x)\varphi(x)&=\int_{-\infty}^{\infty}x\varphi(x)d\delta(x)\\&=x\varphi(x)\delta(x)|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}[\varphi(x)+x\varphi'(x)]\delta(x)dx\\&=\varphi(0)\\&=\int_{-\infty}^{\infty}\varphi(x)\delta(x)dx\end{aligned}$$

得证

> (3)
>
> $$\delta'(-x)=-\delta'(x)$$

$$\begin{aligned}\int_{-\infty}^{\infty}\varphi(x)\delta'(-x)dx&=-\int_{-\infty}^{\infty}\varphi(x)d\delta(-x)\\&=\int_{-\infty}^{\infty}\varphi'(x)\delta(-x)dx\\&=\varphi'(0)\\&=\int_{-\infty}^{\infty}\varphi(x)[-\delta'(x)]dx\end{aligned}$$

得证

2.

> 设有左边变换式$x=x(\xi,\eta),y=y(\xi,\eta),J=\frac{\partial(x,y)}{\partial(\xi,\eta)}$, (x_0,y_0)与(ξ_0,η_0)为相应的点. 证明:
>
> $$\delta(x-x_0,y-y_0)=\frac{1}{|J|}\delta(\xi-\xi_0,\eta-\eta_0)$$
>
> 特别的, 在极坐标情况下, 有$\delta(x-x_0,y-y_0)=\frac{1}{r}\delta(r-r_0,\theta-\theta_0)$.

$$\int_{-\infty}^{\infty}dxdy\ \delta(x-x_0,y-y_0)\varphi(x,y)=\varphi(x_0,y_0)$$

$$\begin{aligned}\int_{-\infty}^{\infty}dxdy\frac{1}{|J|}\delta(\xi-\xi_0,\eta-\eta_0)\varphi(x(\xi,\eta),y(\xi,\eta))&=\int_{-\infty}^{\infty}d\xi d\eta\ \delta(\xi-\xi_0,\eta-\eta_0)\varphi(\xi,\eta)\\&=\varphi(\xi_0,\eta_0)\end{aligned}$$

而(x_0, y_0)和(ξ_0, η_0)为相对应的点, 得证. 极坐标的雅各比数分里算的很明白了, 不写了.

3.

> 把$\delta(x)$在$(-\pi,\pi)$上展开成Fourier级数, 并在弱收敛意义下, 验证所得级数的和确是$\delta(x)$.

$$\delta(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

系数算得

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx)\delta(x)dx = \frac{1}{\pi}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx)\delta(x)dx = 0$$

得到级数为

$$\delta(x) = \frac{1}{2\pi}(1 + 2\sum_{n=1}^{\infty}\cos nx) = \frac{1}{2\pi}\lim_{N\to\infty}\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

弱收敛意义下进行判定. 取任意可积函数$\varphi(x)$,

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}\varphi(x) = \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}[\varphi(x)-\varphi(0)] + \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}\varphi(0)$$

由

$$\lim_{x\to 0}\frac{\varphi(x)-\varphi(0)}{\sin\frac{x}{2}} = \lim_{x\to 0}\frac{\varphi(x)-\varphi(0)}{x/2} = 2\varphi'(0)$$

知$\frac{\varphi(x)-\varphi(0)}{\sin\frac{x}{2}}$可积.

使用黎曼引理, 若$f(x)$可积

$$\lim_{\lambda\to 0}\int_a^b f(x)\sin(\lambda x)dx = 0$$

可知第一项为0.

计算第二项

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}} = \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+2\sum_{n=1}^{N}\cos nx) = 1$$

可知

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}dx\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}\varphi(0) = \varphi(0)$$

得证.

这道题不需要掌握, 考试不会考证明的.

4. 解下列定解问题

> (1)
>
> $$\begin{cases} u_t = a^2 u_{xx}, \quad t > 0, 0 < x, \xi < l \\ u(t, 0) = u(t, l) = 0 \\ u(0, x) = \delta(x - \xi) \end{cases}$$

分离变量, 不多赘述, 解得通解

$$u(t,x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} x \, e^{-\frac{n^2\pi^2a^2}{l^2}t}$$

带入初值条件

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} x = \delta(x - \xi)$$

解得

$$C_n = \frac{2}{l} \sin \frac{n\pi}{l} \xi$$

得到答案

$$u(t,x) = \sum_{n=1}^{\infty} \frac{2}{l} \sin \frac{n\pi}{l} \xi \sin \frac{n\pi}{l} x \, e^{-\frac{n^2\pi^2a^2}{l^2}t}$$

> (2)
>
> $$\begin{cases} u_{tt} = a^2 u_{xx}, \quad t > 0, 0 < x, \xi < l \\ u_x(t, 0) = u_x(t, l) = 0 \\ u(0, x) = 0, \quad u_t(0, x) = \delta(x - \xi) \end{cases}$$

分离变量, 不赘述, 注意$n = 0$

$$u(t,x) = \sum_{n=1}^{\infty} (C_n \sin \frac{n\pi a}{l} t + D_n \cos \frac{n\pi a}{l} t) \cos \frac{n\pi}{l} x + C_0 t + D_0$$

带入初值条件得到

$$D_n = 0 \quad C_0 + \sum_{n=2}^{\infty} C_n \frac{n\pi a}{l} \cos \frac{n\pi a}{l} t \cos \frac{n\pi}{l} x = \delta(x - \xi)$$

解得

$$C_0 = \frac{1}{l} \quad C_n = \frac{2}{n\pi a} \cos \frac{n\pi \xi}{l}$$

得到结果

$$u(t,x) = \frac{t}{l} + \sum_{n=1}^{\infty} \frac{2}{n\pi a} \cos \frac{n\pi}{l} \xi \sin \frac{n\pi a}{l} \cos \frac{n\pi}{l} x$$

5.

> 利用Green第二公式求二维Laplace方程的基本解

参考讲义.

$$U(x,y)=\frac{1}{2\pi}\ln r$$

6. 利用Laplace方程的基本解, 求解下列方程

> (1)
>
> $$\alpha^2u_{xx}+\beta^2u_{yy}=\delta(x,y)\quad x>0,\alpha,\beta<0\text{为常数}$$

换元$\xi=x/\alpha,\eta=y/\beta$

$$u_{\xi\xi}+u_{\eta\eta}=\frac{1}{\alpha\beta}\delta(\xi,\eta)$$

利用二维基本解可得

$$U(x,y)=\frac{1}{4\pi}\ln\left[\left(\frac{x}{\alpha}\right)^2+\left(\frac{y}{\beta}\right)^2\right]$$

> (2)
>
> $$\Delta_2\Delta_2u=\delta(x,y)$$

代入二维基本解

$$\Delta_2u=\frac{1}{2\pi}\ln r$$

由对称性

$$\frac{1}{r}[ru'(r)]'=\frac{1}{2\pi}\ln r$$

基本解只需要一个特解, 得到

$$u(r)=\frac{1}{8\pi}r^2\ln r$$

> (3)
>
> $$\Delta_3\Delta_3u=\delta(x,y)$$

代入三维基本解

$$\Delta_3u=-\frac{1}{4\pi r}$$

同样

$$\frac{1}{r}[ru'(r)]'=-\frac{1}{4\pi r}$$

解得

$$u(r)=-\frac{r}{8\pi}$$