HW 11 solution

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2(1).

记$\bar{u}(p,x)=L[u(y,x)]$, 得: $\frac{d(p\bar{u}-1)}{dx}=\frac{1}{p}$, $\bar{u}|_{x=0}=\bar{f}(p)\Longrightarrow \bar{u}=\frac{x}{p^2}+\bar{f}(p)$,做反变换得:
$u=xy+y+1$

2(2).

做Laplace变换得:

$$\begin{cases} p^2\bar{u}-b=a^2\frac{d^2\bar{u}}{dx^2} \\ \bar{u}|_{x=0}=0,\bar{u}|_{x=+\infty}\text{有界} \end{cases}$$

找到通解:

$$\bar{u}=Ce^{-\frac{p}{a}x}+De^{\frac{p}{a}x}+\frac{b}{p^2},\text{代入边界条件得}:\bar{u}=-\frac{b}{p^2}e^{-\frac{p}{a}x}+\frac{b}{p^2}$$

$$u=L^{-1}[\bar{u}]=bt-b(t-\frac{x}{a})H(t-\frac{x}{a})$$

2(3).

做Laplace变换得:

$$\begin{cases} \frac{d^2\bar{u}}{dt^2}-\frac{p}{a^2}\bar{u}+\frac{u_1}{a^2}=0 \\ \frac{d\bar{u}}{dx}|_{x=0}=0,\bar{u}|_{x=l}=u_0 \end{cases}$$

可求得解:

$$\bar{u}(p,x)=\frac{u_0-u_1}{pch\frac{\sqrt{p}}{a}l}ch\frac{\sqrt{p}}{a}x+\frac{u_1}{p}$$

$$u=L^{-1}[\bar{u}]=u_1+(u_0-u_1)L^{-1}[\frac{u_0-u_1}{pch\frac{\sqrt{p}}{a}l}ch\frac{\sqrt{p}}{a}x]$$

像函数比较复杂，考试不会涉及到，大家可以参考149页的做法。

$ch\frac{\sqrt{p}}{a}l=\cos i\frac{\sqrt{p}}{a}l=0.\Longrightarrow \sqrt{p_k}=\pm i\frac{(2k+1)a\pi}{2l},p_k=-\frac{(2k+1)^2a^2\pi^2}{4l^2},(k=0,1,2,\dots)$

$$\left(pch\frac{\sqrt{p}}{a}l\right)'\Big|_{p=p_k}=-\frac{\pi}{4}(2k+1)\sin\frac{2k+1}{2}\pi=(-1)^{k+1}\frac{\pi}{4}(2k+1)$$

还有:

$$e^{pt}\,\mathrm{ch}\,\frac{\sqrt{p}}{a}x\Big|_{p=p_k}=e^{-\left(\frac{2k+1}{2l}\pi a\right)^2}\cos\frac{(2k+1)\pi}{2l}$$

最后:

$$u(t,x)=u_1+\frac{4}{\pi}(u_1-u_0)\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k+1}e^{-\left(\frac{2k+1}{2l}\pi a\right)^2 t}\cos\frac{(2k+1)\pi x}{2l}$$

2(4).

做Laplace变换得：

$$\begin{cases}\frac{d^2\bar{u}}{dx^2}-\frac{p^2}{a^2}\bar{u}+\frac{p}{a^2(p^2+\omega^2)}=0\\ \bar{u}\big|_{x=0}=0,\quad \bar{u}\big|_{x=+\infty}\text{ 有 界}\end{cases}$$

可以得到解：

$$\bar{u}(p,x)=-\frac{1}{p\left(p^2+\omega^2\right)}e^{-\frac{p}{a}x}+\frac{1}{p\left(p^2+\omega^2\right)}$$

再做反变换：

$$\begin{aligned}u(x,t)&=L^{-1}\left[-\frac{1}{p\left(p^2+\omega^2\right)}e^{-\frac{p}{a}x}\right]+L^{-1}\left[\frac{1}{p\left(p^2+\omega^2\right)}\right]\\&=\frac{2}{\omega^2}\sin^2\frac{\omega t}{2}-\frac{2}{\omega^2}\sin^2\frac{\omega\left(t-\frac{x}{a}\right)}{2}H\left(t-\frac{x}{a}\right)\end{aligned}$$