1. 用Fourier变换解下列定解问题

> 2.
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> $$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(t,x), \quad t>0, -\infty<x<+\infty \\ u(0,x)=0, \quad u_t(0,x)=0 \end{cases}$$

对x进行变换

$$\begin{cases} \partial_t^2 \bar{u} + a^2\lambda^2 \bar{u} = \bar{f}(t,\lambda) \\ \bar{u}(0,\lambda)=0 \quad \bar{u}_t(0,\lambda)=0 \end{cases}$$

对t进行Laplace变换得到

$$\hat{u}(\tau,\lambda) = \frac{\hat{f}(\tau,\lambda)}{\tau^2+a^2\lambda^2}$$

利用卷积公式

$$\begin{aligned} \bar{u}(t,\lambda) &= \frac{\bar{f}(t,\lambda)}{a\lambda} *_t \sin a\lambda t \\ &= \int_0^t \bar{f}(t-\tau,\lambda)\frac{\sin a\lambda\tau}{a\lambda}d\tau \end{aligned}$$

个人习惯, 卷积符号写个下标好区分, 考试不要写. 这个非齐次方程也可以用冲量原理求解.

第一种方法, 拆分$\sin a\lambda t$后利用频移公式

$$\begin{aligned} F^{-1}[\bar{f}(t-\tau,\lambda)\frac{\sin a\lambda\tau}{a\lambda}] &= \frac{1}{2a}\left(F^{-1}[\bar{f}(t-\tau,\lambda)\frac{e^{ia\lambda\tau}}{i\lambda}] - F^{-1}[\bar{f}(t-\tau,\lambda)\frac{e^{-ia\lambda\tau}}{i\lambda}]\right) \\ &= \frac{1}{2a}\left(F^{-1}[\frac{\bar{f}(t-\tau,\lambda)}{i\lambda}](x+a\lambda\tau) - F^{-1}[\frac{\bar{f}(t-\tau,\lambda)}{i\lambda}](x-a\lambda\tau)\right) \\ &= \frac{1}{2a}\left(\int_{-\infty}^{x+a\lambda\tau} f(t-\tau,\xi)d\xi - \int_{-\infty}^{x-a\lambda\tau} f(t-\tau,\xi)d\xi\right) \\ &= \frac{1}{2a}\int_{x-a\lambda\tau}^{x+a\lambda\tau} f(t-\tau,\xi)d\xi \end{aligned}$$

得到结果

$$u(t,x) = \frac{1}{2a}\int_0^t d\tau \int_{x-a\tau}^{x+a\tau} f(t-\tau,\xi)d\xi$$

或者直接进行反变换

$$\begin{aligned} F^{-1}\left[\frac{\sin a\lambda\tau}{\lambda}\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin a\lambda t}{\lambda}\cos\lambda d\lambda \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\frac{\sin\lambda(at+x)}{\lambda} - \frac{\sin\lambda(at-x)}{\lambda}\right]d\lambda \\ &= \frac{1}{4\pi}[\operatorname{sgn}(at+x)+\operatorname{sgn}(at-x)]\int_{-\infty}^{\infty}\frac{\sin\lambda}{\lambda}d\lambda \\ &= \frac{1}{2\pi}H(at-x)H(at+x)\int_{-\infty}^{\infty}\frac{\sin\lambda}{\lambda}d\lambda \end{aligned}$$

sgn为符号函数, H为Heaviside函数.

利用留数定理计算积分.

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}=\operatorname{Im}\pi i\operatorname{Res}(\frac{e^{ix}}{x},0)=\pi$$

得到反变换

$$F^{-1}\left[\frac{\sin a\lambda\tau}{\lambda}\right]=\frac{1}{2}H(at-x)H(at+x)$$

得到答案

$$\begin{aligned}u(t,x)&=\frac{1}{2a}\int_0^t d\tau\int_{-\infty}^{\infty}f(t-\tau,\xi)H(at-x+\xi)H(at+x-\xi)d\xi\\&=\frac{1}{2a}\int_0^t d\tau\int_{x-a\tau}^{x+a\tau}f(t-\tau,\xi)d\xi\end{aligned}$$

> 2.
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> $$\begin{cases}\Delta_2 u=0, & -\infty<x<+\infty,y>0\\ u(x,0)=f(x), & u(x,y)\text{有界}\end{cases}$$

对x进行变换

$$\begin{cases}\partial_y^2\bar{u}-\lambda^2\bar{u}=0\\ \bar{u}(\lambda,0)=\bar{f}(\lambda)\end{cases}$$

由Plancherel公式

$$\langle\bar{u}(\lambda,y),\bar{u}(\lambda,y)\rangle=2\pi\langle u(x,y),u(x,y)\rangle<\infty$$

可知$\bar{u}(\lambda,y)$平方可积. 解得

$$\bar{u}(\lambda,y)=C_1(\lambda)e^{-\lambda y}+C_2(\lambda)e^{\lambda y}$$

可知当$y\to+\infty$, $\lambda>0$时$C_2=0$; $\lambda<0$时$C_1=0$. 可以得到

$$\bar{u}(\lambda,y)=\bar{f}(\lambda)e^{-|\lambda|y}$$

进行反变换

$$\begin{aligned}F^{-1}\left[e^{-|\lambda|y}\right]&=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-|\lambda|y+i\lambda x}d\lambda\\&=\frac{1}{2\pi}\int_0^{\infty}e^{-\lambda y+i\lambda x}d\lambda+\int_{-\infty}^0 e^{\lambda y+i\lambda x}d\lambda\\&=\frac{1}{2\pi}\left(\frac{1}{y+ix}+\frac{1}{y-ix}\right)\\&=\frac{y}{\pi}\frac{1}{y^2+x^2}\end{aligned}$$

则结果为

$$\begin{aligned}u(x,y)&=f(x)*_x\frac{y}{\pi}\frac{1}{y^2+x^2}\\&=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)}{(x-\xi)^2+y^2}d\xi\end{aligned}$$

> 4.
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> $$\begin{cases}\partial_t^2 u+2\partial_t u=\partial_x^2 u-u, & t>0,-\infty<x<\infty\\ u(0,x)=0, & u_t(0,x)=x\end{cases}$$

进行变换

$$\begin{cases}\partial_t^2\bar{u}+2\partial_t\bar{u}=-(\lambda^2+1)\bar{u}\\ u(0,\lambda)=0,\quad u_t(0,x)=\bar{x}\end{cases}$$

目前$f(x)=x$无法直接进行Fourier变换, 需要使用广义函数. 但这并不影响求解, 先设出变换得$\bar{f}=\bar{x}$. 解得

$$\bar{u}(t,\lambda)=e^{-t}\bar{x}\frac{\sin\lambda t}{\lambda}$$

利用上面求得的反变换得到解

$$\begin{aligned}u(t,\lambda)&=e^{-t}x*\frac{1}{2}H(at-x)H(at+x)\\&=e^{-t}\int_{-t}^{t}(x-\xi)d\xi\\&=xte^{-t}\end{aligned}$$

3. 用正弦变换或余弦变换解下列定解问题

> 1.
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> $$\begin{cases}u_t=a^2u_{xx},\quad t>0,x>0,\\ u(t,0)=\varphi(t),\\ u(0,x)=0.\end{cases}$$

使用正弦变换

$$\begin{cases}\partial_t\bar{u}+a^2\lambda^2\bar{u}=a^2\lambda\varphi(t)\\ \bar{u}(0,\lambda)=0\end{cases}$$

解得

$$\bar{u}(t,\lambda)=\int_0^t a^2\lambda\varphi(\tau)e^{-a^2\lambda^2(t-\tau)}d\tau$$

进行反变换

$$\begin{aligned}u(t,x)&=\frac{2a^2}{\pi}\int_0^\infty d\lambda\int_0^t d\tau\ \lambda\varphi(\tau)e^{-a^2\lambda^2(t-\tau)}\sin\lambda x\\&=\frac{2a^2}{\pi}\int_0^t d\tau\ \varphi(\tau)\int_0^\infty d\lambda\ \lambda e^{-a^2\lambda^2(t-\tau)}\sin\lambda x\\&=\frac{a^2}{\pi i}\int_0^t d\tau\ \varphi(\tau)\int_{-\infty}^\infty d\lambda\ \lambda e^{-a^2\lambda^2(t-\tau)+i\lambda x}\\&=\frac{a^2}{\pi i}\int_0^t d\tau\ \varphi(\tau)\int_{-\infty}^\infty d\lambda\ \lambda\exp\left[-(a\sqrt{t-\tau}\lambda-i\frac{x}{2a\sqrt{t-\tau}})^2-\frac{x^2}{4a^2(t-\tau)}\right]\\&=\frac{a^2}{\pi i}\int_0^t d\tau\ \varphi(\tau)e^{-\frac{x^2}{4a^2(t-\tau)}}\{\int_{-\infty}^\infty d\lambda\ (\lambda-i\frac{x}{2a^2(t-\tau)})\exp\left[-(a\sqrt{t-\tau}\lambda-i\frac{x}{2a\sqrt{t-\tau}})^2\right]\\&\quad+\int_{-\infty}^\infty d\lambda\ i\frac{x}{2a^2(t-\tau)}\exp\left[-(a\sqrt{t-\tau}\lambda-i\frac{x}{2a\sqrt{t-\tau}})^2\right]\}\\&=\frac{x}{2\pi a}\int_0^t d\tau\ (t-\tau)^{-\frac{3}{2}}\varphi(\tau)e^{-\frac{x^2}{4a^2(t-\tau)}}\int_{-\infty}^\infty d(a\sqrt{t-\tau}\lambda-i\frac{x}{2a\sqrt{t-\tau}})e^{-(a\sqrt{t-\tau}\lambda-i\frac{x}{2a\sqrt{t-\tau}})^2}\\&=\frac{x}{2a\sqrt{\pi}}\int_0^t d\tau\ (t-\tau)^{-\frac{3}{2}}\varphi(\tau)e^{-\frac{x^2}{4a^2(t-\tau)}}\end{aligned}$$

其中利用了高斯积分.