Preparation

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1 Nabla Operator

1.1 Gradient

$$\nabla u = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial u}{\partial y}\boldsymbol{j} + \frac{\partial u}{\partial z}\boldsymbol{k}$$

1.2 Divergence

$$\nabla \cdot \boldsymbol{E} = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot (E_x\boldsymbol{i} + E_y\boldsymbol{j} + E_z\boldsymbol{k}) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

1.3 Curl

$$\begin{aligned}\nabla \times \boldsymbol{E} &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_y & E_z \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ E_x & E_z \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ E_x & E_y \end{vmatrix}\boldsymbol{k} \\ &= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\boldsymbol{k}\end{aligned}$$

1.4 More formula

(1) $\nabla(u+v) = \nabla u + \nabla v$

(2) $\nabla \cdot (\boldsymbol{E}+\boldsymbol{F}) = \nabla \cdot \boldsymbol{E} + \boldsymbol{\nabla} \cdot \boldsymbol{F}$

(3) $\nabla \times (\boldsymbol{E}+\boldsymbol{F}) = \nabla \times \boldsymbol{E} + \nabla \times \boldsymbol{F}$

(4) $\nabla \cdot (u\boldsymbol{E}) = (\nabla u) \cdot \boldsymbol{E} + u(\nabla \cdot \boldsymbol{E})$

(5) $\nabla \times (u\boldsymbol{E}) = (\nabla u) \times \boldsymbol{E} + u(\nabla \times \boldsymbol{E})$

(6) $\nabla \cdot (\boldsymbol{E} \times \boldsymbol{F}) = \boldsymbol{F} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{F})$

(7) $\nabla \times (\boldsymbol{E} \times \boldsymbol{F}) = (\boldsymbol{F} \cdot \nabla)\boldsymbol{E} - \boldsymbol{F}(\nabla \cdot \boldsymbol{E}) - (\boldsymbol{E} \cdot \nabla)\boldsymbol{F} + \boldsymbol{E}(\nabla \cdot \boldsymbol{F})$

(8) $\nabla(\boldsymbol{E} \cdot \boldsymbol{F}) = (\boldsymbol{F} \cdot \nabla)\boldsymbol{E} + (\boldsymbol{E} \cdot \nabla)\boldsymbol{F} + \boldsymbol{F} \times (\nabla \times \boldsymbol{E}) + \boldsymbol{E} \times (\nabla \times \boldsymbol{F})$

(9) $\nabla \times (\nabla u) = 0$

(10) $\nabla \cdot (\nabla \times \boldsymbol{E}) = 0$
(11) $\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E}$

2 Two functions

2.1 Gamma Function

Def. $\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{dt}$

Properties.

(1). $\Gamma(z+1) = x\Gamma(z)$, for positive integer: $\Gamma(n+1) = n!$

(2). $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(3). $\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$

2.2 Beta Function

Def. $\mathrm{B}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\ \mathrm{d}t$ $\qquad \mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

3 Integral Formula

3.1 Gaussian integral

Basic:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Extension:

$$
\begin{aligned}
&I_{a,0}(\alpha) = \textstyle\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \\
&I_{a,2}(\alpha) = \textstyle\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{-\alpha x^2} x^2 = -\frac{\partial}{\partial \alpha} I_0(\alpha) = \frac{\sqrt{\pi}}{2\alpha^{\frac{3}{2}}} \\
&I_{a,4}(\alpha) = \textstyle\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{-\alpha x^2} x^4 = \left(-\frac{\partial}{\partial \alpha}\right)^2 I_0(\alpha) = \frac{3\sqrt{\pi}}{4\alpha^{\frac{5}{2}}} \\
&\ldots\ldots \\
&I_{a,2n}(\alpha) = \textstyle\int_{-\infty}^{+\infty} \mathrm{d}x \mathrm{e}^{-\alpha x^2} x^{2n} = \left(-\frac{\partial}{\partial \alpha}\right)^n I_0(\alpha) = \frac{(2n-1)!!\sqrt{\pi}}{2^n \alpha^{\frac{2n+1}{2}}}
\end{aligned}
$$

$$\boxed{\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{(2\alpha)^n}, \qquad \int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}}$$

3.2 Exp integral

$$
\begin{aligned}
I_{b,0}(\alpha) &= \int_0^{+\infty} \mathrm{d}x \mathrm{e}^{-\alpha x} = \frac{1}{\alpha} \\
I_{b,1}(\alpha) &= \int_0^{+\infty} \mathrm{d}x \mathrm{e}^{-\alpha x} x = -\frac{\partial}{\partial \alpha} I_0(\alpha) = \frac{1}{\alpha^2} \\
I_{b,2}(\alpha) &= \int_0^{+\infty} \mathrm{d}x \mathrm{e}^{-\alpha x} x^2 = \left(-\frac{\partial}{\partial \alpha}\right)^2 I_0(\alpha) = \frac{2}{\alpha^3} \\
&\cdots \\
&\boxed{I_{b,n}(\alpha) = \int_0^{+\infty} \mathrm{d}x \mathrm{e}^{-\alpha x} x^n = \left(-\frac{\partial}{\partial \alpha}\right)^n I_0(\alpha) = \frac{\Gamma(n+1)}{\alpha^{n+1}}}
\end{aligned}
$$

3.3 Trigonometric integral

(1).

$$
\boxed{
\begin{aligned}
\int x \cos px \ \mathrm{d}x &= \frac{x}{p} \sin px + \frac{1}{p^2} \cos px \\
\int x^2 \cos px dx &= \left(\frac{x^2}{p} - \frac{2}{p^3}\right) \sin px + \frac{2x}{p^2} \cos px \\
\int x \sin px dx &= -\frac{x}{p} \cos px + \frac{1}{p^2} \sin px \\
\int x^2 \sin px dx &= \left(\frac{2}{p^3} - \frac{x^2}{p}\right) \cos px + \frac{2x}{p^2} \sin px
\end{aligned}
}
$$

(2).

$$\forall \alpha, \beta > -1, \int_0^{\frac{\pi}{2}} \cos^\alpha x \sin^\beta x \ \mathrm{d}x = \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right)$$

证明　设$t = \sin^2 x$, 则 $\int_0^{\frac{\pi}{2}} \cos^\alpha x \sin^\beta x \ \mathrm{d}x = \frac{1}{2} \int_0^1 t^{\frac{\beta-1}{2}} (1-t)^{\frac{\alpha-1}{2}} \ \mathrm{d}t.$

$$\int_0^{\pi/2} \sin^\alpha \theta \mathrm{d}\theta = \int_0^{\pi/2} \cos^\alpha \theta \mathrm{d}\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{\alpha+1}{2}\right)}{2\Gamma\left(1+\frac{\alpha}{2}\right)}$$

$\alpha = 2n$:

$$\int_0^{\pi/2} \sin^{2n} \theta \mathrm{d}\theta = \int_0^{\pi/2} \cos^{2n} \theta \mathrm{d}\theta = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}$$

$\alpha = 2n+1$:

$$\int_0^{\pi/2} \sin^{2n+1} \theta \mathrm{d}\theta = \int_0^{\pi/2} \cos^{2n+1} \theta \mathrm{d}\theta = \frac{(2n)!!}{(2n+1)!!}$$

3.4 Exp&Trigonometric

$$\begin{aligned}\int_0^{\infty} e^{-ax^2}\cos(bx)\mathrm{d}x &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-ax^2}\cos(bx)\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-ax^2}e^{ibx}\ \mathrm{d}x \\ &= \frac{1}{2}e^{-b^2/4a}\int_{-\infty}^{\infty} e^{-a(x-ib/2a)^2}\ \mathrm{d}t \\ &= \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-b^2/4a}\end{aligned}$$

$$\boxed{\int_0^{\infty} e^{-ax^2}\cos(bx)\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-b^2/4a}}$$

4 Series

(1).sum

$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$

$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3 = \frac{1}{4}n^2(n+1)^2$

$\sum_{k=1}^{n} k^4 = 1^4+2^4+3^4+\cdots+n^4 = \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right)$

(2).Infinite Series:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln 2, \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}, \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}, \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

5 欧拉方程

$$r^2R'' + prR' + qR = 0$$

做变换 $r = e^s$ 可化为：

$$\frac{d^2R}{ds^2} + (p-1)\frac{dR}{ds} + qR = 0$$

坐标系	直角坐标系	圆柱坐标系	球坐标系
u_1, u_2, u_3	x,y,z	ρ, ϕ, z	r,θ, ϕ
h_1	1	1	1
h_2	1	ρ	r
h_3	1	1	$\mathrm{rsin}\,\theta$

表 1: 拉梅系数

6 拉梅系数

$$\nabla\varphi = \frac{1}{h_1}\frac{\partial\varphi}{\partial u_1}\vec{e}_1 + \frac{1}{h_2}\frac{\partial\varphi}{\partial u_2}\vec{e}_2 + \frac{1}{h_3}\frac{\partial\varphi}{\partial u_3}\vec{e}_3$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_3h_1A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$\nabla\times\vec{A} = \begin{vmatrix} \frac{\vec{e}_1}{h_2h_3} & \frac{\vec{e}_2}{h_3h_1} & \frac{\vec{e}_3}{h_1h_2} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\nabla^2\varphi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u_3}\right)\right]$$